Question:

Using the shortcuts, find the derivative of $f(x) = \frac{(2x-1)^2}{\sqrt{x}}$ at $x=4$.

Solution: $f(x) = \frac{4x^2-4x+1}{\sqrt{x}} = \frac{4x^2-4x+1}{x^{1/2}}$

$= (4x^2-4x+1)\cdot \frac{1}{x^{1/2}} = (4x^2-4x+1)\cdot x^{-1/2}$

$= 4x^{3/2} - 4x^{1/2} + x^{-1/2}$

$f'(x) = [4x^{3/2} - 4x^{1/2} + x^{-1/2}]'$

$= [4x^{3/2}]' - [4x^{1/2}]' + [x^{-1/2}]'$

$= 4[x^{3/2}]' - 4[x^{1/2}]' + [x^{-1/2}]'$

$= 4\cdot\frac{3}{2}x^{\frac{3}{2}-1} - 4\cdot\frac{1}{2}x^{\frac{1}{2}-1} + \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$ $\quad [x^n]' = nx^{n-1}$

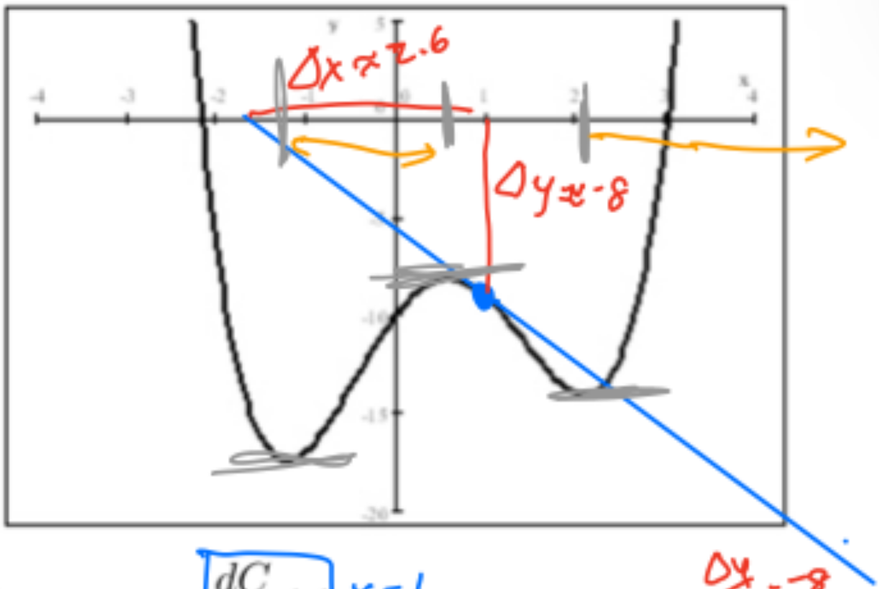
$= 6x^{1/2} - 2x^{-1/2} - \frac{1}{2}x^{-3/2}$

$(x=4)$

$= 6\cdot 2 - 2\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{8} = 12 - 1 - \frac{1}{16} = 11 - \frac{1}{16}$

$= \boxed{\frac{175}{16}}$.

3.21 Consider the graph of $y = C(x)$ below.



(a) Estimate $\boxed{\dfrac{dC}{dx}(1)}$. $x=1$ tangent line slope $\frac{\Delta y}{\Delta x} = \frac{-8}{2.6} \approx \boxed{-3}$.

(b) Find all numbers a such that $C'(a) = 0$.

(c) Find the interval(s) of a such that $C'(a) > 0$.

Our New Trick:

The product rule:

$$[f(x)\cdot g(x)]' = f'(x)\cdot g(x) + f(x)g'(x)$$

Proof of the product rule:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$[f(x)\cdot g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \underbrace{\left(\frac{f(x+h) - f(x)}{h}\right)}_{\to f'(x)} \underbrace{g(x+h)}_{\to g(x)} + f(x)\underbrace{\frac{(g(x+h) - g(x))}{h}}_{\to g'(x)}$$

$$= \boxed{f'(x)g(x) + f(x)g'(x)}. \blacksquare$$

Technical facts we are using:

① We're are assuming both f & g are differentiable (meaning derivative limit exists)

② If $g(x)$ is differentiable at c, then $g(x)$ is also continuous at c.

[Why? If $\lim_{h\to 0} \frac{g(c+h) - g(c)}{h} = g'(c)$ exists, then $g(c+h) - g(c)$ must go to zero, because $h \to 0$. $\lim_{h\to 0} g(c+h) = g(c)$ is the same thing as saying g is continuous at c.]

Examples of using the product rule:

① Find the derivative of $x^2\sin(x)$.

$$[\underbrace{x^2}_{f(x)}\underbrace{\sin(x)}_{g(x)}]' = f'(x)g(x) + f(x)g'(x)$$

$$= \boxed{2x\sin(x) + x^2\cdot\cos(x).}$$

② $(\sqrt{x}\,e^x)'$ at $x=1$.

$$= \left(\tfrac{1}{2}x^{-1/2}\right)\cdot e^x + x^{1/2}\cdot e^x$$

$$= \left(\tfrac{1}{2}x^{-1/2} + x^{1/2}\right)e^x = \left(\tfrac{1}{2}\cdot 1 + 1\right)e^1$$

$$= \boxed{\tfrac{3}{2}e}.$$